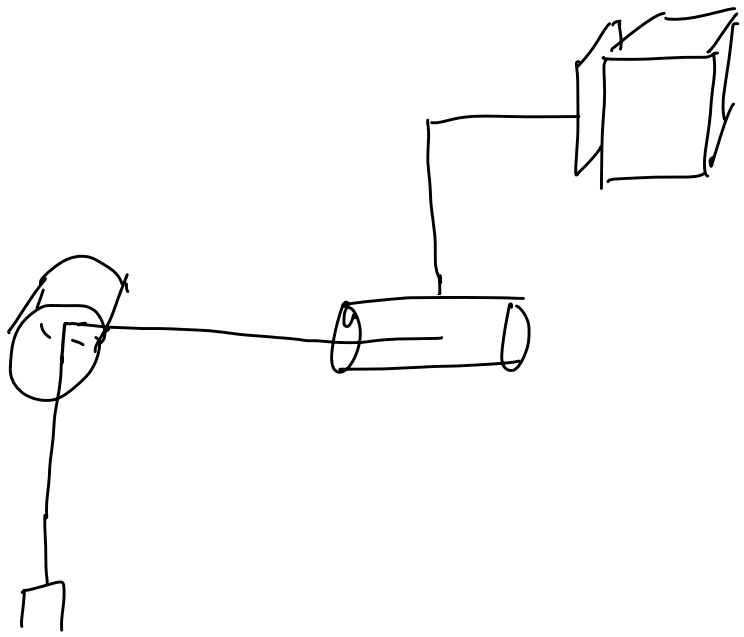
TASKS



Task 1:

a) Draw the workspace for the manipulator

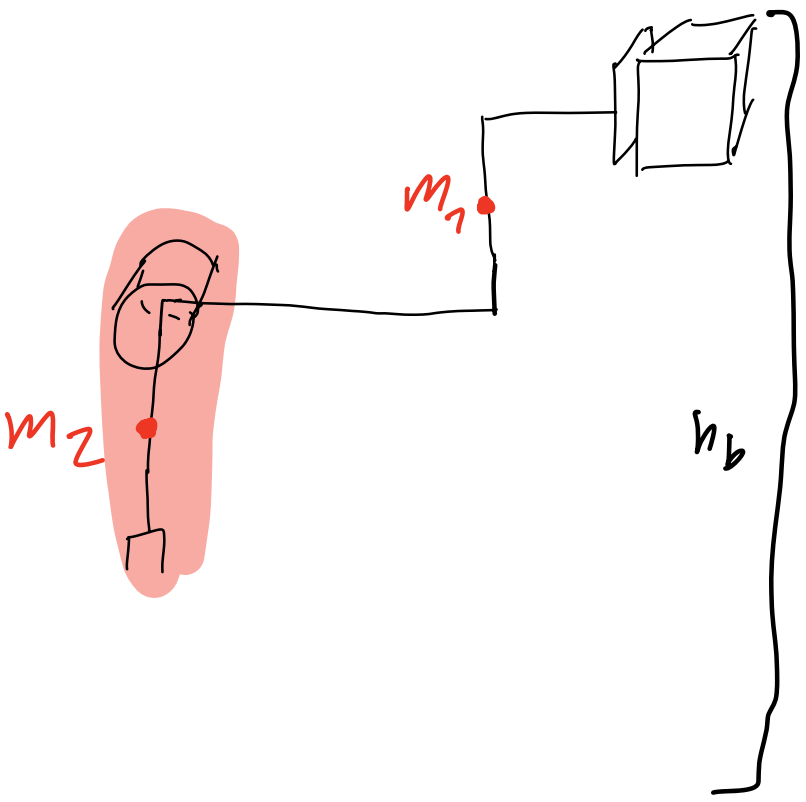
b) Assign coordinate frames in accordance with the Denavit-Hartenberg convention. Also, find the DH-parameters and place them into a tab

c) Derive the forward kinematics for the manipulator

d) Derive the inverse kinematics for the manipulator

e) Derive the Jacobian matrix for the manipulator

Task 2



a) Find the symbolic expression for the system's potential energy

b) Find the symbolic expression for the system's kinetic energy given $I_2 = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$, where m_1 is a point mass and m_2 is an uniformly distributed mass

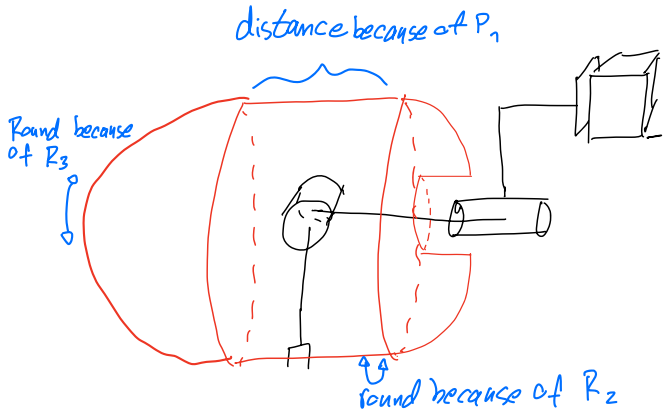
c) What would be the difference in the expression for the kinetic energy if m_1 was an uniformly distributed mass?

d) Assemble the Langrangian L and derive the dynamics of the manipulator using the Euler-Langrange formulation, write the equation of motion on the form

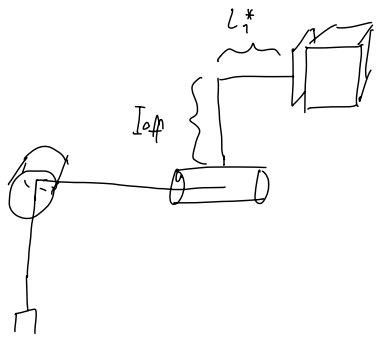
$$D(q)\ddot{q} + C(q,\dot{q})\dot{q} + g(q) = \tau$$

SOLUTION 1

a) workspace:



draw inner

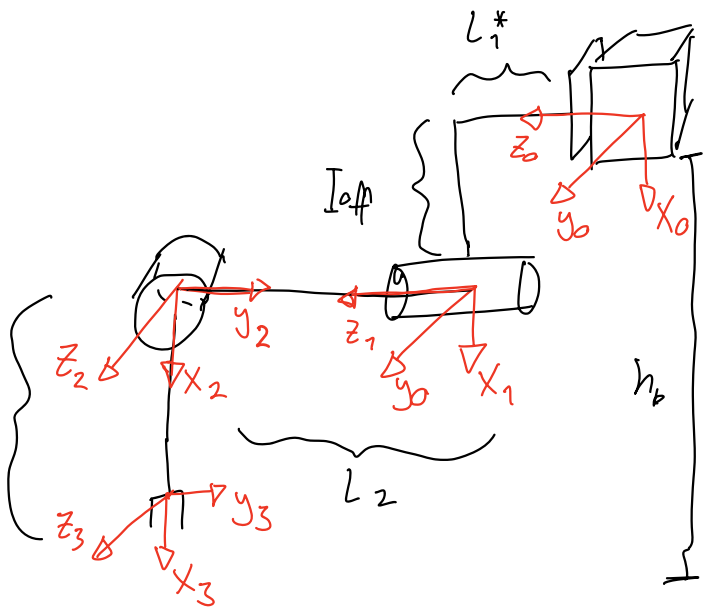


b) DH-table:

coordinate frames:

DH

Link	a	α	d	θ
1	I_{off}	0	L_1^*	0
2	0	-90°	L_2	θ_2^*
3	L_3	0	0	θ_3^*



c) forward Kinematics:

$$T_1^0 = \begin{bmatrix} 1 & 0 & 0 & I_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^1 = \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_3^2 = \begin{bmatrix} C_3 & -S_3 & 0 & L_3C_3 \\ S_3 & C_3 & 0 & L_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_1^0 T_2^1 = T_2^0 = \begin{bmatrix} 1 & 0 & 0 & I_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_2 & 0 & -S_2 & 0 \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & L_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & -S_2 & I_{off} \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & L_2+L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$T_2^0 T_3^2 = T_3^0 = \begin{bmatrix} C_2 & 0 & -S_2 & I_{off} \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & L_2+L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_3 & -S_3 & 0 & L_3C_3 \\ S_3 & C_3 & 0 & L_3S_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_2C_3 & -C_2S_3 & -S_2 & C_2C_3L_3 + I_{off} \\ S_2C_3 & -S_2S_3 & C_2 & S_2C_3L_3 \\ -S_3 & -C_3 & 0 & -S_3L_3 + L_2 + L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

inverse kinematics

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} C_2 C_3 L_3 + I_{0A} \\ S_2 C_3 L_3 \\ -S_3 L_3 + L_2 + L_1^* \end{bmatrix}$$

Position of tooltip

from last column in forward

We must find $L_1^*, \theta_2^*, \theta_3^*$:

$$\frac{y}{x_c - I_{0A}} = \frac{S_2 C_3 L_3}{C_2 C_3 L_3} \Rightarrow \theta_2^* = \operatorname{atan2}\left(\frac{y}{x_c - I_{0A}}\right)$$

$$y_c = S_2 C_3 L_3$$

$$C_3 = \frac{y_c}{S_2 L_3} := D \qquad C_3 = \sqrt{1 - D^2}$$

$$\theta_3^* = \operatorname{atan2}\left(\frac{D}{\sqrt{1-D^2}}\right)$$

$$z_c = -S_3 L_3 + L_2 + L_1^*$$

$$L_1^* = z_c + S_3 L_3 - L_2$$

d) inverse kinematics (alternative with explicit expressions)

$$x = L_{off} + L_3 c_2 c_3 \quad \Rightarrow x - L_{off} = L_3 c_2 c_3$$

$$y = L_3 s_2 c_3$$

- Try to remove θ_2 by squaring and adding:

$$(x - l_{off})^2 + y^2 = (L_3 c_2 c_3)^2 + (L_3 s_2 c_3)^2 = L_3^2 c_2^2 c_3^2 + L_3^2 s_2^2 c_3^2$$

$$= L_3^2 (c_2^2 + s_2^2) c_3^2 = L_3^2 c_3^2$$

$$\Rightarrow c_3^2 = \frac{(x - l_{off})^2 + y^2}{L_3^2} \Rightarrow c_3 = \pm \frac{\sqrt{(x - l_{off})^2 + y^2}}{L_3}$$

$$c_3^2 + s_3^2 = 1 \Rightarrow s_3 = \pm\sqrt{1 - c_3^2} = \pm\sqrt{1 - \frac{(x - l_{off})^2 + y^2}{L_3^2}}$$

$$\theta_3^* = \operatorname{atan2}(s_3, c_3) = \operatorname{atan2}\left(\pm\sqrt{1 - \frac{(x - l_{off})^2 + y^2}{L_3^2}}, \pm\frac{\sqrt{(x - l_{off})^2 + y^2}}{L_3}\right)$$

$$= \operatorname{atan2}\left(\pm\sqrt{L_3^2 - (x - l_{off})^2 - y^2}, \pm\sqrt{(x - l_{off})^2 + y^2}\right)$$

$$z = L_1^* + L_2 - L_3 s_3 \Rightarrow L_3 s_3 = L_1^* + L_2 - z$$

Similar to $L_3 c_3$

- Add square of this to $(x - l_{off})^2 + y^2$

$$(x - l_{off})^2 + y^2 + (L_1^* + L_2 - z)^2 = L_3^2 c_3^2 + L_3^2 s_3^2 = L_3^2$$

$$\Rightarrow (L_1^* + L_2 - z)^2 = L_3^2 - (x - l_{off})^2 - y^2$$

$$L_1^* + L_2 - z = \pm\sqrt{L_3^2 - (x - l_{off})^2 - y^2}$$

$$L_1^* = z - L_2 \pm \sqrt{L_3^2 - (x - l_{off})^2 - y^2}$$

e) Jacobian:

from forward

$$T_1^0 = \begin{bmatrix} 1 & 0 & 0 & I_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow O_1^0,\ Z_1^0$$

$$T_2^0 = \begin{bmatrix} C_2 & 0 & -S_2 & I_{off} \\ S_2 & 0 & C_2 & 0 \\ 0 & -1 & 0 & L_2+L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow O_2^0,\ Z_2^0$$

$$T_3^0 = \begin{bmatrix} C_2C_3 & -C_2S_3 & -S_2 & C_2C_3L_3+I_{off} \\ S_2C_3 & -S_2S_3 & C_2 & S_2C_3L_3 \\ -S_3 & -C_3 & 0 & -S_3L_3+L_2+L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \rightarrow O_3^0,\ Z_3^0$$

$$Z_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad O_0 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$J = \begin{bmatrix} Z_0 & Z_1 \times (O_3 - O_1) & Z_2 \times (O_3 - O_2) \\ 0 & Z_1 & Z_2 \end{bmatrix}$$

$$Z_1 \times (O_3 - O_1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} C_2C_3L_3 + I_{off} \\ S_2C_3L_3 \\ -S_3L_3 + L_2 + L_1^* \end{bmatrix} - \begin{bmatrix} I_{off} \\ 0 \\ L_1^* \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} C_2C_3L_3 \\ S_2C_3L_3 \\ -S_3L_3 + L_2 \end{bmatrix} = \begin{bmatrix} -S_2C_3L_3 \\ C_2C_3L_3 \\ 0 \end{bmatrix}$$

$\langle a, b, c \rangle = \langle 0, 0, 1 \rangle$

$= \begin{bmatrix} -b \\ a \\ c \end{bmatrix}$

$$Z_2 \times (O_3 - O_2) = \begin{bmatrix} -S_2 \\ C_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} C_2C_3L_3 + I_{off} \\ S_2C_3L_3 \\ -S_3L_3 + L_2 + L_1^* \end{bmatrix} - \begin{bmatrix} I_{off} \\ 0 \\ L_1^* + L_2 \end{bmatrix}$$

$$= \begin{bmatrix} -S_2 \\ C_2 \\ 0 \end{bmatrix} \times \begin{bmatrix} C_2C_3L_3 \\ S_2C_3L_3 \\ -S_3L_3 \end{bmatrix} = \begin{bmatrix} -C_2S_3L_3 \\ -S_2S_3L_3 \\ -S_2^2C_3L_3 - C_2^2C_3L_3 \end{bmatrix}$$

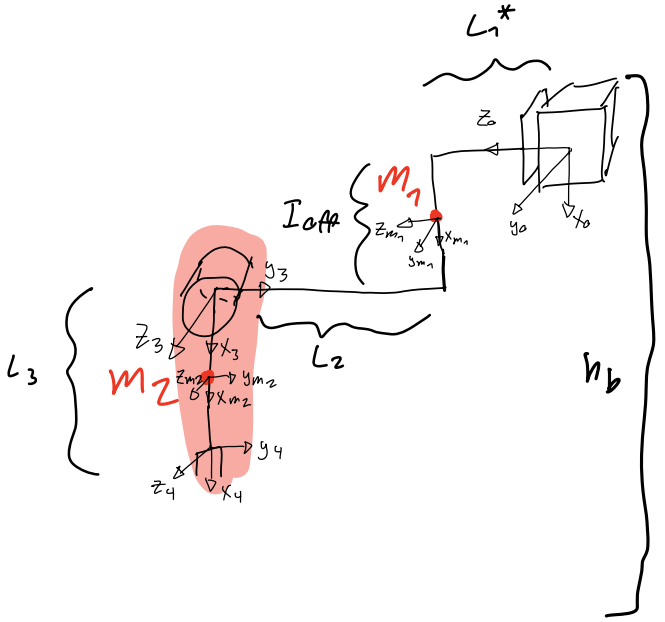
$$= \begin{bmatrix} -C_2S_3L_3 \\ -S_2S_3L_3 \\ -C_3L_3 \end{bmatrix}$$

$\langle -S_2, C_2, 0 \rangle \times \langle a, b, c \rangle$

$= \begin{bmatrix} C_2c \\ S_2c \\ -S_2b - C_2a \end{bmatrix}$

$$J = \begin{bmatrix} 0 & -S_2C_3L_3 & -C_2S_3L_3 \\ 0 & C_2C_3L_3 & -S_2S_3L_3 \\ 1 & 0 & -C_3L_3 \\ 0 & 0 & -S_2 \\ 0 & 0 & C_2 \\ 0 & 1 & 0 \end{bmatrix}$$

SOLUTION 2



a) Potential energy:

$$P = P_1 + P_2$$

$$P_1 = m_1 g h_1$$
$$= m_1 g (h_b - \tfrac{1}{2} I_{off})$$

$$P_2 = m_2 g h_2$$
$$= m_2 g (h_b - C_3 \tfrac{L_3}{2} - I_{off})$$

$$P = m_1 g h_b - \tfrac{1}{2} m g I_{off} + m_2 g h_b - m_2 g C_3 \tfrac{L_3}{2} - m_2 g I_{off}$$

from forward

$$T_1^0 = \begin{bmatrix} 1 & 0 & 0 & I_{off} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ the height must be in x-direction

$$T_3^0 = \begin{bmatrix} C_2C_3L_3 + I_{off} \\ S_2C_3L_3 \\ -S_3L_3 + L_2 + L_1^* \end{bmatrix}$$ I remember to divide I_{off} by 2 for m_1 and L_3 by 2 for m_2

$C_2 = 1, S_2 = 0$ $\quad \theta_2 = 0$

$$T_3^0 = \begin{bmatrix} C_3L_3 + I_{off} \\ 0 \\ -S_3L_3 + L_2 + L_1^* \end{bmatrix}$$

b) Kinetic energy

$$K = K_1 + K_2$$

$$K_1 = \tfrac{1}{2} m_1 V_1^T V_1 + \underbrace{\tfrac{1}{2} m_1 w_1^T R_1 I R_1^T w_1}_{=0 \text{ because it is point mass}}$$

$$= \tfrac{1}{2} m_1 \begin{bmatrix} 0 & 0 & \dot{L}_1^* \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{L}_1^* \end{bmatrix} = \tfrac{1}{2} m_1 \dot{L}_1^{*2}$$

$$J = \begin{bmatrix} 0 & -S_2C_3L_3 & -C_2S_3L_3 \\ 0 & C_2C_3L_3 & -S_2S_3L_3 \\ 1 & 0 & -C_3L_3 \\ 0 & 0 & -S_2 \\ 0 & 0 & C_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$J_{m_2} \cdot \dot{q}_{m_2} = J \cdot \begin{bmatrix} \dot{L}_1^* \\ 0 \\ \dot{\theta}_3^* \end{bmatrix} = \begin{bmatrix} 0 + 0 & -S_3 \frac{L_3}{2} \dot{\theta}_3^* \\ 0 + 0 + 0 \\ \dot{L}_1^* + 0 & -C_3 \frac{L_3}{2} \dot{\theta}_3^* \\ 0 + 0 + 0 \\ 0 + 0 + \dot{\theta}_3^* \\ 0 + 0 + 0 \end{bmatrix}$$

$$J_{m_1} \cdot \dot{q}_{m_1} = J \cdot \begin{bmatrix} \dot{L}_1^* \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 + 0 + 0 \\ 0 + 0 + 0 \\ \dot{L}_1^* + 0 + 0 \\ 0 + 0 + 0 \\ 0 + 0 + 0 \\ 0 + 0 + 0 \end{bmatrix}$$

v w

$$\Rightarrow R_{m_2}^0 = \begin{bmatrix} C_3 & -S_3 & 0 \\ 0 & 0 & 1 \\ -S_3 & -C_3 & 0 \end{bmatrix}$$

$R_{m_2} = R_3^0$ where $\theta_2 = 0$ because Joint 3 and m_2 has the same orientation

from forward

$$T_3^0 = \begin{bmatrix} C_2C_3 & -C_2S_3 & -S_2 & C_2C_3L_3 + I_{off} \\ S_2C_3 & -S_2S_3 & C_2 & S_2C_3L_3 \\ -S_3 & -C_3 & 0 & -S_3L_3 + L_2 + L_1^* \\ 0 & 0 & 0 & 1 \end{bmatrix} \Rightarrow R_{m_2}^0 = \begin{bmatrix} C_3 & -S_3 & 0 \\ 0 & 0 & 1 \\ -S_3 & -C_3 & 0 \end{bmatrix}$$

$$K_2 = \frac{1}{2} m_2 v_2^T v_2 + \frac{1}{2} \quad w_2^T R_2 I R_2^T w_2$$

$$= \frac{1}{2} m_2 \left[-s_3 \frac{l_3}{2} \dot{\theta}_3^* \quad 0 \quad \dot{l}_1^* - c_3 \frac{l_3}{2} \dot{\theta}_3^* \right] \begin{bmatrix} -s_3 \frac{l_3}{2} \dot{\theta}_3^* \\ 0 \\ \dot{l}_1^* - c_3 \frac{l_3}{2} \dot{\theta}_3^* \end{bmatrix}$$

$$\begin{bmatrix} c_3 & 0 & s_3 \\ 0 & 0 & c_3 \\ 0 & -1 & 0 \end{bmatrix}$$

$$+ \frac{1}{2} \quad \begin{bmatrix} 0 & \dot{\theta}_3^* & 0 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 \\ 0 & 0 & 1 \\ -s_3 & -c_3 & 0 \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} c_3 & 0 & -s_3 \\ -s_3 & 0 & -c_3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta}_3^* \\ 0 \end{bmatrix}$$

$$= \frac{1}{2} m_2 \left(s_3^2 \frac{l_3^2}{4} \dot{\theta}_3^{*2} + \dot{l}_1^{*2} + c_3^2 \frac{l_3^2}{4} \dot{\theta}_3^{*2} - 2 \dot{l}_1^* c_3 \frac{l_3}{2} \dot{\theta}_3^* \right)$$

$$+ \frac{1}{2} \quad \begin{bmatrix} 0 & 0 & \dot{\theta}_3^* \end{bmatrix} \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_3^* \end{bmatrix}$$

$$= \frac{1}{2} m_2 \frac{l_3^2}{4} \dot{\theta}_3^{*2} + \frac{1}{2} m_2 \dot{l}_1^{*2} - \frac{1}{2} m_2 \dot{l}_1^* c_3 l_3 \dot{\theta}_3^* + \frac{1}{2} \quad I_{zz} \dot{\theta}_3^{*2}$$

$$K = \frac{1}{2} m_1 \dot{l}_1^{*2} + \frac{1}{2} m_2 \frac{l_3^2}{4} \dot{\theta}_3^{*2} + \frac{1}{2} m_2 \dot{l}_1^{*2} - \frac{1}{2} m_2 \dot{l}_1^* c_3 l_3 \dot{\theta}_3^* + \frac{1}{2} \quad I_{zz} \dot{\theta}_3^{*2}$$

c) Langrangian:

$$L = K - P$$

$$L = \frac{1}{2} m_1 \dot{l}_1^{*2} + \frac{1}{2} m_2 \frac{L_3^2}{4} \dot{\theta}_3^{*2} + \frac{1}{2} m_2 \dot{L}_1^{*2} - \frac{1}{2} m_2 \dot{l}_1^{*} C_3 L_3 \dot{\theta}_3^{*} + \frac{1}{2} I_{zz} \dot{\theta}_3^{*2}$$
$$- m_1 g h_b + \frac{1}{2} m g I_{aH} - m_2 g h_b + m_2 g C_3 \frac{L_3}{2} + m_2 g I_{aH}$$

$$\frac{\partial}{\partial L_1^{*}} = 0$$

$$\frac{\partial}{\partial \dot{L}_1^{*}} = m_1 \dot{L}_1^{*} + m_2 \dot{L}_1^{*} - \frac{1}{2} m_2 C_3 L_3 \dot{\theta}_3^{*}$$

$$\frac{\partial}{\partial t} \frac{\partial}{\partial \dot{L}_1^{*}} = m_1 \ddot{L}_1^{*} + m_2 \ddot{L}_1^{*} + \frac{1}{2} m_2 S_3 L_3 \dot{\theta}_3^{2*} - \frac{1}{2} m_2 C_3 L_3 \ddot{\theta}_3^{*}$$

$$\frac{\partial}{\partial \theta_3^{*}} = -\frac{1}{2} m_2 \dot{l}_1^{*} S_3 L_3 \dot{\theta}_3^{*} - m_2 g S_3 \frac{L_3}{2}$$

$$\frac{\partial}{\partial \dot{\theta}_3^{*}} = m_2 \frac{L_3^2}{4} \dot{\theta}_3^{*} - \frac{1}{2} m_2 \dot{l}_1^{*} C_3 L_3 + I_{zz} \dot{\theta}_3^{*}$$

$$\frac{d}{dt} \frac{\partial}{\partial \dot{\theta}_3^{*}} = m_2 \frac{L_3^2}{4} \ddot{\theta}_3^{*} - \frac{1}{2} m_2 \ddot{L}_1^{*} C_3 L_3 + \frac{1}{2} m_2 \dot{l}_1^{*} S_3 L_3 \dot{\theta}_3^{*} + I_{zz} \ddot{\theta}_3^{*}$$

$$L = \begin{bmatrix} m_1 + m_2 & -\frac{1}{2} m_2 C_3 L_3 \\ \frac{1}{2} m_2 C_3 L_3 & I_{zz} + m_2 \frac{L_3^2}{4} \end{bmatrix} \begin{bmatrix} \ddot{l}_1^{*} \\ \ddot{\theta}_3^{*} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{1}{2} m_2 s_3 L_3 \dot{\theta}_3^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{L}_1^* \\ \dot{\theta}_3^* \end{bmatrix}$$

$$+ \begin{bmatrix} 0 \\ m_2 g s_3 \frac{L_3}{2} \end{bmatrix}$$

$$V_2 = \begin{bmatrix} -\frac{L_3}{2} s_3 \dot{\theta}_3 \\ 0 \\ \dot{L}_1^* - \frac{L_3}{2} c_3 \dot{\theta}_3 \end{bmatrix}$$

$$V_2^T V_2 = |V_2|^2 = \left(-\frac{L_3^2}{2} s_3 \dot{\theta}_3\right)^2 + \left(\dot{L}_1^* - \frac{L_3}{2} c_3 \dot{\theta}_3\right)^2$$

$$= \frac{L_3^2}{4} s_3^2 \dot{\theta}_3^2 + \dot{L}_1^{*2} - \dot{L}_1^* L_3 c_3 \dot{\theta}_3 + \frac{L_3^2}{4} c_3 \dot{\theta}_3^2$$

$$= \frac{L_3^2}{4} \dot{\theta}_3^2 + \dot{L}_1^{*2} - \dot{L}_1^* L_3 c_3 \dot{\theta}_3$$

$$\Rightarrow \frac{1}{2} m_2 V_2^T V_2 = \frac{1}{2} m_2 \left(\frac{L_3^2}{4} \dot{\theta}_3^2 + \dot{L}_1^{*2} - \dot{L}_1^* L_3 c_3 \dot{\theta}_3\right)$$

$$= \frac{1}{8} m_2 L_3^2 \dot{\theta}_3^2 + \frac{1}{2} m_2 \dot{L}_1^{*2} - \frac{1}{2} m_2 \dot{L}_1^* L_3 c_3 \dot{\theta}_3$$

$$\tau = \begin{bmatrix} m_1 \ddot{L}_1^* + m_2 \ddot{L}_1^* + \frac{1}{2} m_2 s_3 L_3 \dot{\theta}_3^{2*} - \frac{1}{2} m_2 c_3 L_3 \ddot{\theta}_3^* \\ m_2 \frac{L_3^2}{4} \ddot{\theta}_3^* + \frac{1}{2} m_2 \ddot{L}_1^* c_3 L_3 - \frac{1}{2} m_2 \dot{L}_1^* s_3 L_3 \dot{\theta}_3^* + \frac{1}{2} m_2 \dot{L}_1^* s_3 L_3 \dot{\theta}_3^* + m_2 g s_3 \frac{L_3}{2} + I_{zz_3} \ddot{\theta}_3^* \end{bmatrix}$$

$$= \begin{bmatrix} m_1 + m_2 & -\frac{1}{2} m_2 c_3 L_3 \\ \frac{1}{2} m_2 c_3 L_3 & I_{zz} + \frac{1}{4} m_3 L_3^2 \end{bmatrix} \begin{bmatrix} \ddot{L}_1^* \\ \ddot{\theta}_3^* \end{bmatrix} + \begin{bmatrix} 0 & \frac{1}{2} m_2 s_3 L_3 \dot{\theta}_3^* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{L}_1^* \\ \dot{\theta}_3 \end{bmatrix} + \begin{bmatrix} 0 \\ m_2 g s_3 \frac{L_3}{2} \end{bmatrix}$$

$$\tau_1 = m_1 \ddot{L}_1^* + m_2 \ddot{L}_2^* - \frac{1}{2} m_2 L_3 c_3 \ddot{\theta}_3^* + \frac{1}{2} m_2 L_3 s_3 \dot{\theta}_3^2$$

$$\tau_2 = I_{zz} \ddot{\theta}_3^* + \frac{1}{4} m_2 L_3^2 \ddot{\theta}_3^* - \frac{1}{2} m_2 L_2 c_3 \ddot{\theta}_1 + \frac{1}{2} m_2 L_3 g s_2$$